Nonclassical Electromagnetic Dynamics

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Abstract: Our paper is concerned with effects of special forces on the motion of particles. §1(§2) studies the single-time geometric dynamics induced by electromagnetic vector fields (1-forms) and by the Euclidean structure of the space. §3 defines the second-order forms or vectors. §4 (§5) describes a nonclassical electric (magnetic) dynamics produced by an ”electric (magnetic) second-order Lagrangian”, via the extremals of the energy functional. §6 generalize this dynamics for a general second-order Lagrangian, having in mind possible applications for dynamical systems coming from Biomathematics, Economical Mathematics, Industrial Mathematics, etc.

Key–Words: geometric dynamics, second-order vectors, second-order Lagrangian, nonclassical dynamics.

1 Single-time geometric dynamics induced by electromagnetic vector fields

Let U be a domain of linear homogeneous isotropic media in the Riemannian manifold $(M = R^3, \delta_{ij})$. Maxwell's equations (coupled PDEs of first order)

div $D = \rho$, rot $H = J + \partial_t D$, ∂_t = time derivative operator

div $B = 0$, rot $E = -\partial_t B$

with the constitutive equations

$$B = \mu H, \quad D = \varepsilon E,$$

on $R \times U$, reflect the relations between the electromagnetic fields:

E	$[V/m]$	electric field strength
H	$[A/m]$	magnetic field strength
J	$[A/m^2]$	electric current density
ε	$[As/Vm]$	permitivity
μ	$[Vs/Am]$	permeability
B	$[T] = [Vs/m^2]$	magnetic induction (magnetic flux density)
D	$[C/m^2] = [As/m^2]$	electric displacement (electric flux density)

Since div $B = 0$, the vector field B is source free, hence may be expressed as rot of some vector potential A, i.e., $B =$ rot A. Then the electric field strength is $E = -\text{grad}\, V - \partial_t A$.

It is well-known that the motion of a charged particle in the electromagnetic fields is described by the ODE system (*Lorentz World-Force Law*)

$$m\frac{d^2x}{dt^2} = e\left(E + \frac{dx}{dt} \times B\right),$$

$$x = (x^1, x^2, x^3) \in U \subset R^3,$$

where m is the *mass*, and e is the *charge* of the particle. Of course, these are Euler-Lagrange equations produced by the Lorentz Lagrangian

$$L_1 = \frac{1}{2}m\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + e\delta_{ij}\frac{dx^i}{dt}A^j - eV.$$

The associated Lorentz Hamiltonian is

$$H_1 = \frac{m}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + eV.$$

A similar mechanical motion was discovered by Popescu [7], accepting the existence of a *gravitovortex field* represented by the force $G + \dfrac{dx}{dt} \times \Omega$, where G is the gravitational field and Ω is a vortex determining the gyroscopic part of the force. The papers [5], [8], [9], [12], [14], [15] confirm the point of view of Popescu via geometric dynamics.

1.1 Single-time geometric dynamics produced by vector potential A

To conserve the traditional formulas, we shall refer to field lines of the vector potential ”$-A$” using dimensional homogeneous relations. These curves are solutions of the ODE system

$$m\frac{dx}{dt} = -eA.$$

This ODE system and the Euclidean metric produce the least squares Lagrangian

$$L_2 = \frac{1}{2}\delta_{ij}\left(m\frac{dx^i}{dt} + eA^i\right)\left(m\frac{dx^j}{dt} + eA^j\right)$$

$$= \frac{1}{2}\left\|m\frac{dx}{dt} + eA\right\|^2.$$

The Euler-Lagrange equations associated to L_2 are

$$m\frac{d^2x^i}{dt^2} = e\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right)\frac{dx^j}{dt} + \frac{e^2}{m}\frac{\partial f_A}{\partial x^i} - e\partial_t A,$$

where

$$f_A = \frac{1}{2}\delta_{ij}A^iA^j$$

is the energy *density* produced by A. Equivalently, it appears a *single-time geometric dynamics*

$$m\frac{d^2x}{dt^2} = e\frac{dx}{dt} \times B + \frac{e^2}{m}\nabla f_A - e\partial_t A, \quad B = \text{rot } A,$$

a motion in a gyroscopic force [12], [14], [15] or a B-vortex dynamics. The associated Hamiltonian is

$$H_2 = \frac{1}{2}\delta_{ij}\left(m\frac{dx^i}{dt} - eA^i\right)\left(m\frac{dx^j}{dt} + eA^j\right)$$

$$= \frac{m^2}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - e^2 f_A.$$

Remark. Generally, the single-time geometric dynamics produced by the vector potential "$-A$" is different from the classical Lorentz World-Force Law because

$$L_2 - mL_1 = \frac{1}{2}e^2\delta_{ij}A^iA^j + meV$$

and the force $\frac{e}{m}\nabla f_A - \partial_t A$ is not the electric field $E = -\nabla V - \partial_t A$. In other words the Lagrangians L_1 and L_2 are not in the same equivalence class of Lagrangians.

1.2 Single-time geometric dynamics produced by magnetic induction B

Since we want to analyze the geometric dynamics using units of measure, the magnetic flow must be described by

$$m\frac{dx}{dt} = \lambda B,$$

where the unit measure for the constant λ is $[kgm^3/Vs^2]$.

The associated least squares Lagrangian is

$$L_3 = \frac{1}{2}\delta_{ij}\left(m\frac{dx^i}{dt} - \lambda B^i\right)\left(m\frac{dx^j}{dt} - \lambda B^j\right)$$

$$= \frac{1}{2}\left\|m\frac{dx}{dt} - \lambda B\right\|^2.$$

This gives the Euler-Lagrange equations (*single-time magnetic geometric dynamics*)

$$m\frac{d^2x}{dt^2} = \lambda\frac{dx}{dt} \times \text{rot } B + \frac{\lambda^2}{m}\nabla f_B + \lambda\partial_t B,$$

where

$$f_B = \frac{1}{2}\delta_{ij}B^iB^j = \frac{1}{2}||B||^2$$

is the *magnetic energy density*. This is in fact a dynamics under a gyroscopic force or in J-vortex.

The associated Hamiltonian is

$$H_3 = \frac{1}{2}\delta_{ij}\left(m\frac{dx^i}{dt} - \lambda B^i\right)\left(m\frac{dx^j}{dt} + \lambda B^j\right)$$

$$= \frac{m^2}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - \lambda^2 f_B.$$

1.3 Single-time geometric dynamics produced by electric field E

The electric flow is described by

$$m\frac{dx}{dt} = \lambda E,$$

where the unit measure of the constant λ is $[kgm^2/Vs]$. It appears the least squares Lagrangian

$$L_4 = \frac{1}{2}\delta_{ij}\left(m\frac{dx^i}{dt} - \lambda E^i\right)\left(m\frac{dx^j}{dt} - \lambda E^j\right)$$

$$= \frac{1}{2}\left\|m\frac{dx}{dt} - \lambda E\right\|^2,$$

with Euler-Lagrange equations (*single-time electric geometric dynamics*)

$$m\frac{d^2x}{dt^2} = \lambda\frac{dx}{dt} \times \text{rot } E + \frac{\lambda^2}{m}\nabla f_E + \lambda\partial_t E,$$

where

$$f_E = \frac{1}{2}\delta_{ij}E^iE^j = \frac{1}{2}||E||^2$$

is the *electric energy density*; here we have in fact a dynamics in $\partial_t B$-vortex. The associated Hamiltonian is

$$H_4 = \frac{1}{2}\delta_{ij}\left(m\frac{dx^i}{dt} - \lambda E^i\right)\left(m\frac{dx^j}{dt} + \lambda E^j\right)$$
$$= \frac{m^2}{2}\delta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - \lambda^2 f_E.$$

Open problem. As is well-known, charged particles in the magnetic field of the earth spiral from pole to pole. Similar motions are also observed in laboratory plasmas and inferred for electrons in metal subjected to an external magnetic field. Till now, these motions where justified by the classical Lorentz World-Force Law. Can we justify such motions using the geometric dynamics produced by suitable vector fields in the sense of present paper and the papers [5], [7]-[12], [14], [15]?

2 Potentials associated to electromagnetic forms

Let $U \subset R^3 = M$ be a domain of linear homogeneous isotropic media. Which mathematical object shall we select to model the electromagnetic fields; vector fields or forms? It was shown [1] that, the magnetic induction B, the electric displacement D, and the electric current density J are all 2-forms; the magnetic field H, and the electric field E are 1-forms; the electric charge density ρ is a 3-form. The operator d is the exterior derivatives and the operator ∂_t is the time derivative.

In terms of differential forms, the *Maxwell equations* on $U \times R$ can be expressed as

$$dD = \rho, \qquad dH = J + \partial_t D$$
$$dB = 0, \qquad dE = -\partial_t B.$$

The constitutive relations are

$$D = \varepsilon * E,\ B = \mu * H,$$

where the star operator $*$ is the Hodge operator, ε is the permitivity, and μ is the scalar permeability.

The local components E_i, $i = 1, 2, 3$, of the 1-form E are called *electric potentials*, and the local components H_i, $i = 1, 2, 3$, of the 1-form H are called *magnetic potentials*. Since the electric field E, and the magnetic field H are 1-forms [1], in the Sections 4-5 we combine our ideas [8]-[16] with the ideas of Emery [3] and Foster [4], creating a nonclassical electric or magnetic dynamics. Finally, we generalize the results to nonclassical dynamics induced by a second-order Lagrangian (Section 6).

2.1 Potential associated to electric 1-form E

Let us consider the function $V : R \times U \to R$, $(t, x) \to V(t, x)$ and the Pfaff equation $dV = -E$ or the equivalent PDE system $\frac{\partial V}{\partial x^i} = -E_i$, $i = 1, 2, 3$. Of course, the complete integrability conditions require $dE = 0$ (*electrostatic field*), which is not always satisfied. In any situation we can introduce the least squares Lagrangian

$$L_9 = \frac{1}{2}\delta_{ij}\left(\frac{\partial V}{\partial x^i} + E_i\right)\left(\frac{\partial V}{\partial x^j} + E_j\right)$$
$$= \frac{1}{2}||dV + E||^2.$$

These produce the Euler-Lagrange equation (Poisson equation)

$$\Delta V = -\text{div}\, E,$$

and consequently V must be the *electric potential*. For a linear isotropic material, we have $D = \varepsilon E$, with $\rho = \text{div}\, D = \varepsilon \text{div}\, E$. We get the potential equation for a homogeneous material (ε = constant)

$$\Delta V = -\frac{\rho}{\varepsilon}.$$

For the charge free space, we have $\rho = 0$, and then the potential V satisfies the Laplace equation

$$\Delta V = 0$$

(*harmonic function*). The Lagrangian L_9 produces the Hamiltonian

$$H_9 = \frac{1}{2}(dV - E, dV + E),$$

and the momentum-energy tensor field

$$T^i{}_j = \frac{\partial V}{\partial x^j}\frac{\partial L}{\partial\left(\frac{\partial V}{\partial x^i}\right)} - L_9\delta^i_j$$
$$= \frac{\partial V}{\partial x^j}\left(\frac{\partial V}{\partial x^i} + E_i\right) - L_9\delta^i_j.$$

2.2 Potential associated to magnetic 1-form H

Now, we consider the Pfaff equation $d\varphi = H$ or the PDE system $\frac{\partial \varphi}{\partial x^i} = H_i$, $i = 1, 2, 3$, $\varphi : R \times U \to R$, $(t, x) \to \varphi(t, x)$. The complete integrability conditions $dH = 0$ are satisfied only for particular cases. If we build the least squares Lagrangian

$$L_{10} = \frac{1}{2}\delta^{ij}\left(\frac{\partial \varphi}{\partial x^i} - H_i\right)\left(\frac{\partial \varphi}{\partial x^j} - H_j\right)$$

$$= \frac{1}{2}||d\varphi - H||^2,$$

then we obtain the Euler-Lagrange equation (Laplace equation) and consequently φ must be the *magnetic potential*. The Lagrangian L_{10} produces the Hamiltonian $H_{10} = \frac{1}{2}(d\varphi - H, d\varphi + H)$.

2.3 Potential associated to 1-form potential A

Since $dB = 0$, there exists an 1-form potential A satisfying $B = dA$. Now let us consider the Pfaff equation $d\psi = A$ or the equivalent PDEs system $\frac{\partial \psi}{\partial x^i} = A_i$, $i = 1, 2, 3$, $\psi : R \times U \to R$, $(t, x) \to \psi(t, x)$. The complete integrability conditions $dA = 0$ are satisfied only for $B = 0$. If we build the least squares Lagrangian

$$L_{11} = \frac{1}{2}\delta^{ij}\left(\frac{\partial \psi}{\partial x^i} - A_i\right)\left(\frac{\partial \psi}{\partial x^j} - A_j\right)$$

$$= \frac{1}{2}||d\psi - A||^2,$$

then we obtain the Euler-Lagrange equation (Laplace equation) $\Delta\psi = 0$. The Lagrangian L_{11} gives the Hamiltonian $H_{11} = \frac{1}{2}(d\psi - A, d\psi + A)$ and the momentum-energy tensor field

$$T^i{}_j = \frac{\partial \psi}{\partial x^j}\left(\frac{\partial \psi}{\partial x^i} - A_i\right) - L_{11}\delta^i_j.$$

Open problem. Find interpretations for the extremals of least squares Lagrangians

$$L_{12} = \frac{1}{2}||dA - B||^2$$

$$= \frac{1}{2}\delta^{ik}\delta^{jl}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} - B_{ij}\right)\left(\frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^l} - B_{kl}\right)$$

$$L_{13} = \frac{1}{2}||dE + \partial_t B||^2 + \frac{1}{2}||dH - J - \partial_t D||^2$$

$$+\frac{1}{2}||dD - \rho||^2 + \frac{1}{2}||dB||^2,$$

which are not solutions of Maxwell equations.

Remarks. 1) There are a lot of applications of previous type in applied sciences. One of the most important is in the *material strength*. In problems associated with the torsion of a cylinder or prism, one has to investigate the functional

$$J(z(\cdot)) = \int_D \left(\left(\frac{\partial z}{\partial x} - y\right)^2 + \left(\frac{\partial z}{\partial y} + x\right)^2\right) dxdy$$

for an extremum. The Euler-Lagrange equation, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, shows that the extremals are harmonic functions. Of course, J has no global ”minimum point” since the PDE system $\frac{\partial z}{\partial x} = y$, $\frac{\partial z}{\partial y} = -x$ is not completely integrable.

2) All previous examples in this section are *single-component potentials*. Similarly we can introduce the *multi-component potentials*.

3 Second-Order Forms and Vectors

Now we will concentrate on certain geometric ideas that are very important in the physical and stochastic applications. To avoid too much repetition, M will denote a differentiable manifold of dimension n, and all the functions are of class C^∞.

Let $x^i = x^i(x^{i'})$, $i, i' = 1, \ldots, n$ be a changing of coordinates on M. Then we introduce the symbols

$$D^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}, \quad D^i_{i'j'} = \frac{\partial^2 x^i}{\partial x^{i'} x^{j'}}.$$

For a differentiable function $f : M \to R$ we use the simplified coordinate expression $f(x^{i'}) = f(x^i(x^{i'}))$, the first order derivatives $f_{,i'}, f_{,i}$ and the second order derivatives $f_{,i'j'}, f_{,ij}$. These are connected by the rule

$$(f_{,i'}, f_{,i'j'}) = (f_{,i}, f_{,ij})\begin{pmatrix} D^i_{i'} & D^i_{i'j'} \\ 0 & D^i_{i'}D^j_{j'} \end{pmatrix}. \quad (1)$$

The pair (first-order partial derivatives, second-order partial derivatives) possesses a ”tensorial” change law that the second derivative, by itself, lacks. This pair was used in the classical works like ”contact element” or like ”jet”.

If a curve is given by the parametric equations $x^i = x^i(t)$, $t \in I$, then the preceding diffeomorphism modifies the pair $(\ddot{x}, \dot{x} \otimes \dot{x})^T$ as follows

$$\dot{x}^{i'} = \dot{x}^i D^{i'}_i, \ \ddot{x}^{i'} = \ddot{x}^i D^{i'}_i + \dot{x}^i\dot{x}^j D^{i'}_{ij},$$

$$\begin{pmatrix} \ddot{x}^{i'} \\ \dot{x}^{i'}\dot{x}^{j'} \end{pmatrix} = \begin{pmatrix} D^{i'}_i & D^{i'}_{ij} \\ 0 & D^{i'}_i D^{j'}_j \end{pmatrix}\begin{pmatrix} \ddot{x}^i \\ \dot{x}^i\dot{x}^j \end{pmatrix}.$$

The pair (acceleration, ”square of velocity”) is suggested by the equations of geodesics

$$\ddot{x}^k(t) + \Gamma^k_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t) = 0.$$

Consequently, when first and second derivatives come into play together, then matrices of blocks such as

$$K = \begin{pmatrix} D^i_{i'} & D^i_{i'j'} \\ 0 & D^i_{i'}D^j_{j'} \end{pmatrix}, \; K^{-1} = \begin{pmatrix} D^{i'}_i & D^{i'}_{ij} \\ 0 & D^{i'}_i D^{j'}_j \end{pmatrix}$$

are useful for changing the components of pairs of objects.

Definition. Any pair (ω_i, ω_{ij}) admitting the changing law (1), where (ω_i) is an 1-form and (ω_{ij}) is symmetric, will be called a *second-order form* on M. A *second-order vector field* on M is a second-order differential operator, with no constant term.

A second-order form can be written as $\theta = \theta_i d^2x^i + \theta_{ij} dx^i \otimes dx^j$, where the components θ_i and $\theta_{ij} = \theta_{ji}$ are smooth functions. A second-order vector field writes $Xf = \ell^{ij} D_{ij} f + \ell^i D_i f$, where the components $\ell^{ij} = \ell^{ji}$ and ℓ^i are smooth functions. The theory of second-order vectors or forms appears in Emery [3], with applications in stochastic problems, and in Foster [4], suggesting new point of view about the fields theory. We use these ideas to define meaningful second-order Lagrangians (kinetic potentials) and to study their extremals.

4 Dynamics induced by second-order electric form

Let E_i be the electric potentials. The usual derivative $E_{i,j}$ may be decomposed into skew-symmetric and symmetric parts,

$$E_{i,j} = \frac{1}{2}(E_{i,j} - E_{j,i}) + \frac{1}{2}(E_{i,j} + E_{j,i}).$$

The skew-symmetric part (vortex)

$$m_{ij} = \frac{1}{2}(E_{i,j} - E_{j,i})$$

is called *Maxwell tensor field* giving the opposite of the time derivative of magnetic induction. The symmetric part

$$\frac{1}{2}(E_{i,j} + E_{j,i})$$

is not an ordinary tensor, but the pair

$$(E_i, \; \frac{1}{2}(E_{i,j} + E_{j,i}))$$

is a second-order object [3]-[4].

Let ω_{ij} be a general object such that (E_i, ω_{ij}) is a second-order object. The difference

$$(0, \omega_{ij} - \frac{1}{2}(E_{i,j} + E_{j,i}))$$

is a second-order object determined by $g_{ij} = \omega_{ij} - \frac{1}{2}(E_{i,j} + E_{j,i})$. If ω_{ij} is symmetric, then g_{ij} is a symmetric tensor field; if we add $\det(g_{ij}) \neq 0$, then g_{ij} can be used as a semi-Riemann metric. In this context the equality

$$(E_i, \omega_{ij}) = (E_i, \frac{1}{2}(E_{i,j} + E_{j,i})) + (0, g_{ij})$$

shows that the valuable objects

$$\frac{1}{2}(E_{i,j} + E_{j,i})$$

come from electricity and mate with gravitational potentials g_{ij}. Consequently they have to be *electrogravitational potentials*.

The preceding potentials determine an *electric energy second-order Lagrangian*,

$$L_{el} = E_i(x(t), t) \, \frac{d^2x^i}{dt^2}(t)$$

$$+ \frac{1}{2}(E_{i,j} + E_{j,i})(x(t), t) \frac{dx^i}{dt}(t) \frac{dx^j}{dt}(t).$$

If the *most general energy second-order Lagrangian* is given by

$$L_{ge} = E_i(x(t), t) \frac{d^2x^i}{dt^2}(t) + \omega_{ij}(x(t), t) \frac{dx^i}{dt}(t) \frac{dx^j}{dt}(t),$$

and the *gravitational energy first-order Lagrangian* is

$$L_g = g_{ij}(x(t), t) \frac{dx^i}{dt}(t) \frac{dx^j}{dt}(t),$$

then $L_{ge} = L_{el} + L_g$.

To simplify, let us take $E = E(x)$. Since the distribution generated by the electric 1-form E is given by the Pfaff equation $E_i(x) dx^i = 0$, the electric energy Lagrangian is zero along integral curves of this distribution. Also the general energy second-order Lagrangian for $E = E(x), \omega_{ij} = \omega_{ij}(x)$ determines the *energy functional*

$$\int_a^b \left(E_i(x(t)) \, \frac{d^2x^i}{dt^2} + \omega_{ij}(x(t)) \frac{dx^i}{dt}(t) \frac{dx^j}{dt}(t) \right) dt. \tag{2}$$

We denote

$$\omega_{ijk} = \frac{1}{2}(\omega_{kj,i} + \omega_{ki,j} - \omega_{ij,k})$$

(the Christoffel symbols of ω_{ij}),

$$E_{ijk} = \frac{1}{2} \left(E_{k,ij} + E_{j,ik} - E_{i,jk} \right).$$

Theorem. *The extremals of the energy functional (2) are described by the Euler-Lagrange ODEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (\omega_{ijk} - E_{ijk})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$x(a) = x_a, x(b) = x_b.$$

(geodesics with respect to an Otsuki connection [6]).

Proof. Using the second-order Lagrangian

$$L = E_i\,\frac{d^2x^i}{dt^2} + \omega_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt},$$

the Euler-Lagrange equations

$$L_{x^k} - \frac{d}{dt}L_{\frac{dx^k}{dt}} + \frac{d^2}{dt^2}L_{\frac{d^2x^k}{dt^2}} = 0$$

transcribe like the equations in the theorem.

To a Lagrangian there may corresponds a field theory. Consequently we obtain a field theory having as basis the general electrogravitational potentials. The pure gravitational potentials are given by

$$g_{ij} = \omega_{ij} - \frac{1}{2}(E_{i,j} + E_{j,i}).$$

We define $\Gamma_{ijk} = \omega_{ijk} - E_{ijk}$. It is verified that $\Gamma_{ijk} = g_{ijk} + m_{ijk}$, where g_{ijk} are the Christoffel symbols of g_{ij}, and $m_{ijk} = m_{ij,k} + m_{ik,j}$ is the symmetrized derivative of the Maxwell tensor

$$m_{ij} = \frac{1}{2}(E_{i,j} - E_{j,i}).$$

Corollary. *The extremals of the energy functional (1) are described by the Euler-Lagrange ODEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (g_{kji} + m_{kji})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$x(a) = x_a, x(b) = x_b.$$

(geodesics with respect to an Otsuki connection [6]).

5 Dynamics induced by second-order magnetic form

Let H_i be the magnetic potentials. The usual derivative $H_{i,j}$ may be decomposed into skew-symmetric and symmetric parts,

$$H_{i,j} = \frac{1}{2}(H_{i,j} - H_{j,i}) + \frac{1}{2}(H_{i,j} + H_{j,i}),$$

where

$$M_{ij} = \frac{1}{2}(H_{i,j} - H_{j,i})$$

is the *Maxwell tensor field* (vortex) giving the sum between the electric current density and the time derivative of the electric displacement. The pair

$$\left(H_i,\ \frac{1}{2}(H_{i,j} + H_{j,i})\right)$$

is a second-order object. If (H_i, ω_{ij}) is a general second-order object, then the difference

$$g_{ij} = \omega_{ij} - \frac{1}{2}(H_{i,j} + H_{j,i})$$

represents the gravitational potentials (a metric) provided that ω_{ij} is symmetric, g_{ij} is a (0,2) tensor field and $\det(g_{ij}) \neq 0$. Consequently the valuable objects $\frac{1}{2}(H_{i,j} + H_{j,i})$, which come from magnetism and mate, have to be *magnetogravitational potentials*.

The preceding potentials produce the following energy Lagrangians:

1) *magnetic energy second-order Lagrangian*,

$$L_{ma} = H_i(x(t), t)\,\frac{d^2x^i}{dt^2}(t)$$

$$+\frac{1}{2}(H_{i,j} + H_{j,i})(x(t), t)\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t);$$

2) *gravitational energy first-order Lagrangian*,

$$L_g = g_{ij}(x(t), t)\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t);$$

3) *general energy second-order Lagrangian*,

$$L_{ge} = H_i(x(t), t)\,\frac{d^2x^i}{dt^2} + \omega_{ij}(x(t), t)\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t).$$

These satisfy the relation $L_{ge} = L_{ma} + L_g$.

To simplify, let us take $H = H(x)$. Since the distribution generated by the magnetic 1-form H is given by the Pfaff equation $H_i(x)dx^i = 0$, the magnetic energy Lagrangian is zero along integral curves of this distribution. Also the general energy second-order Lagrangian for $H = H(x), \omega_{ij} = \omega_{ij}(x)$ determines the *energy functional*

$$\int_a^b \left(H_i(x(t))\,\frac{d^2x^i}{dt^2}(t) + \omega_{ij}(x(t))\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t)\right)dt \qquad (3)$$

determined by a second-order Lagrangian which is linear in acceleration.

We denote

$$\omega_{ijk} = \frac{1}{2}(\omega_{kj,i} + \omega_{ki,j} - \omega_{ij,k})$$

(the Christoffel symbols of ω_{ij}),

$$H_{ijk} = \frac{1}{2}\left(H_{k,ij} + H_{j,ik} - H_{i,jk}\right).$$

Theorem. *The extremals of the energy functional (3) are solutions of the Euler-Lagrange ODEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (\omega_{ijk} - H_{ijk})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$x(a) = x_a, x(b) = x_b.$$

(geodesics with respect to an Otsuki connection [6]).

To an energy Lagrangian there may corresponds a field theory. Consequently we obtain a field theory having as basis the general magnetogravitational potentials. The pure gravitational potentials are (components of a Riemann or semi-Riemann metric)

$$g_{ij} = \omega_{ij} - \frac{1}{2}(H_{i,j} + H_{j,i}).$$

We introduce $\Gamma_{ijk} = \omega_{ijk} - H_{ijk}$. It is verified the relation $\Gamma_{ijk} = g_{ijk} + M_{ijk}$, where g_{ijk} are the Christoffel symbols of g_{ij}, and $M_{ijk} = M_{ij,k} + M_{ik,j}$ is the symmetrized derivative of the Maxwell tensor

$$M_{ij} = \frac{1}{2}(H_{i,j} - H_{j,i}).$$

Corollary. *The extremals of the energy functional (3) are solutions of the Euler-Lagrange ODEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (g_{kji} + M_{kji})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$x(a) = x_a, x(b) = x_b.$$

(geodesics with respect to an Otsuki connection [6]).

6 Dynamics induced by second-order general form

Now we want to extend the preceding explanations since they can be applied to dynamical systems coming from Biomathematics, Economical Mathematics, Industrial Mathematics etc.

Let ω_i be given potentials (given form). The usual partial derivative $\omega_{i,j}$ may be decomposed as

$$\omega_{i,j} = \frac{1}{2}(\omega_{i,j} - \omega_{j,i}) + \frac{1}{2}(\omega_{i,j} + \omega_{j,i}),$$

where

$$\mathcal{M}_{ij} = \frac{1}{2}(\omega_{i,j} - \omega_{j,i})$$

is a *Maxwell tensor field* (vortex). The pair

$$\left(\omega_i, \frac{1}{2}(\omega_{i,j} + \omega_{j,i})\right)$$

is a second-order form. If (ω_i, ω_{ij}) is a general second-order form, then we suppose that the difference

$$g_{ij} = \omega_{ij} - \frac{1}{2}(\omega_{i,j} + \omega_{j,i})$$

represents the components of a metric, i.e., ω_{ij} is symmetric, g_{ij} is a (0,2) tensor field and $\det(g_{ij}) \neq 0$.

The preceding potentials produce the following energy Lagrangians:

1) *potential-produced energy Lagrangian,*

$$L_{pp} = \omega_i(x(t), t)\,\frac{d^2x^i}{dt^2}(t)$$

$$+\frac{1}{2}(\omega_{i,j} + \omega_{j,i})(x(t), t)\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t);$$

2) *gravitational energy Lagrangian,*

$$L_g = g_{ij}(x(t), t)\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t);$$

3) *general energy Lagrangian,*

$$L_{ge} = \omega_i(x(t), t)\,\frac{d^2x^i}{dt^2} + \omega_{i,j}(x(t), t)\frac{dx^i}{dt}\frac{dx^j}{dt};$$

which verify

$$L_{ge} = L_{pp} + L_g.$$

The Pfaff equation $\omega_i(x)dx^i = 0$, $i = 1, \ldots, n$ defines a distribution on M. The valuable objects

$$\frac{1}{2}(\omega_{i,j} + \omega_{j,i})$$

are the components of the second fundamental form of that distribution [6]. The potential-produced energy Lagrangian is zero along integral curves of the distribution generated by the given 1-form $\omega = (\omega_i(x))$.

In the autonomous case, the general energy Lagrangian produces the *energy functional*

$$\int_a^b \left(\omega_i(x(t))\,\frac{d^2x^i}{dt^2}(t) + \omega_{ij}(x(t))\frac{dx^i}{dt}(t)\frac{dx^j}{dt}(t)\right) dt. \tag{4}$$

This energy functional is associated to a particular Lagrangian L of order two.

We denote

$$\omega_{ijk} = \frac{1}{2}(\omega_{kj,i} + \omega_{ki,j} - \omega_{ij,k})$$

(the Christoffel symbols of ω_{ij}),

$$\Omega_{ijk} = \frac{1}{2}\left(\omega_{k,ij} + \omega_{j,ik} - \omega_{i,jk}\right).$$

Theorem. *The extremals of the energy functional (4) are solutions of the Euler-Lagrange ODEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (\omega_{ijk} - \Omega_{ijk})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$x(a) = x_a, x(b) = x_b$$

(geodesics with respect to an Otsuki connection [6]).

To an energy Lagrangian there may correspond a field theory. We introduce $\Gamma_{ijk} = \omega_{ijk} - \Omega_{ijk}$. After some computations we find $\Gamma_{ijk} = g_{ijk} + \mathrm{m}_{ijk}$, where g_{ijk} are the Christoffel symbols of g_{ij}, and

$$\mathrm{m}_{ijk} = \mathcal{M}_{ij,k} + \mathcal{M}_{ik,j}$$

is the symmetrized derivative of the Maxwell tensor field $\mathcal{M}$.

Corollary. *The extremals of the energy functional (4) are solutions of the Euler-Lagrange ODEs*

$$g_{ki}\frac{d^2x^i}{dt^2} + (g_{kji} + m_{kji})\frac{dx^i}{dt}\frac{dx^j}{dt} = 0,$$

$$x(a) = x_a, x(b) = x_b$$

(geodesics with respect to an Otsuki connection [6]).

Open problems: (1) Find the linear connections in the sense of Crampin [2] associated to the preceding second-order ODEs. (2) Analyze the second variations of the preceding energy functionals. (2) Analyze the symmetries of the preceding second order differential systems (see also [16]). (3) Find practical interpretations for motions known as geometric dynamics, gravitovortex motion and second-order force motion (see also [5]).

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