

Necessary Optimality Conditions for Fractional Action-Like Problems with Intrinsic and Observer Times

GASTÃO S. F. FREDERICO

University of Cape Verde
Department of Science and Technology
Praia, Santiago
CAPE VERDE

DELFIN F. M. TORRES

University of Aveiro
Department of Mathematics
3810-193 Aveiro
PORTUGAL

Abstract: We prove higher-order Euler-Lagrange and DuBois-Reymond stationary conditions to fractional action-like variational problems. More general fractional action-like optimal control problems are also considered.

Key-Words: calculus of variations, FALVA problems, higher-order Euler-Lagrange equations, higher-order DuBois-Reymond stationary condition, multi-time control theory.

1 Introduction

The study of fractional problems of the calculus of variations and respective Euler-Lagrange type equations is a subject of strong current research because of its numerous applications: see e.g. [1, 3, 4, 6, 7, 10, 12, 13, 14]. F. Riewe [14] obtained a version of the Euler-Lagrange equations for problems of the calculus of variations with fractional derivatives, that combines both conservative and non-conservative cases. In 2002 O. P. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [1]. Then, these Euler-Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [3]. In [12] fractional problems of the calculus of variations with symmetric fractional derivatives are considered and correspondent Euler-Lagrange equations obtained, using both Lagrangian and Hamiltonian formalisms. In all the above mentioned studies, Euler-Lagrange equations depend on left and right fractional derivatives, even when the problem depend only on one type of them. In [13] problems depending on symmetric derivatives are considered for which Euler-Lagrange equations include only the derivatives that appear in the formulation of the problem. In [4, 8] Euler-Lagrange type equations for problems of the calculus of variations which depend on the Riemann-Liouville derivatives of order (α, β) , $\alpha > 0$, $\beta > 0$, are studied.

In [5, 16, 17], C. Udriste and his coauthors remark that the standard multi-variable variational calculus has some limitations which the multi-time control theory successfully overcomes. For instance, the

classical multi-variable variational calculus cannot be applied directly to create a multi-time maximum principle. In [6, 7] two-time Riemann-Liouville fractional integral functionals, depending on a parameter α but not on fractional-order derivatives of order α , are introduced and respective fractional Euler-Lagrange type equations obtained. In [11], Jumarie uses the variational calculus of fractional order to derive an Hamilton-Jacobi equation, and a Lagrangian variational approach to the optimal control of one-dimensional fractional dynamics with fractional cost function. Here, we extend the Euler-Lagrange equations of [6, 7] by considering two-time fractional action-like variational problems with higher-order derivatives. A DuBois-Reymond stationary condition is also proved for such problems. Finally, we study more general two-time optimal control type problems.

2 Preliminaries

In 2005, El-Nabulsi (cf. [6]) introduced the following Fractional Action-Like VARIational (FALVA) problem.

Problem 1 Find the stationary values of the integral functional

$$I[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\theta, q(\theta), \dot{q}(\theta)) (t-\theta)^{\alpha-1} d\theta \quad (P_1)$$

under given initial conditions $q(a) = q_a$, where $\dot{q} = \frac{dq}{d\theta}$, Γ is the Euler gamma function, $0 < \alpha \leq 1$, θ is the intrinsic time, t is the observer time, $t \neq \theta$, and the

Lagrangian $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^2 -function with respect to all its arguments.

Along all the work, we denote by $\partial_i L$ the partial derivative of a function L with respect to its i -th argument, $i \in \mathbb{N}$.

Next theorem summarizes the main result of [6].

Theorem 2 (cf. [6]) *If $q(\cdot)$ is a solution of Problem 1, that is, $q(\cdot)$ offers a stationary value to functional (P_1) , then $q(\cdot)$ satisfies the following Euler-Lagrange equations:*

$$\begin{aligned} \partial_2 L(\theta, q(\theta), \dot{q}(\theta)) - \frac{d}{d\theta} \partial_3 L(\theta, q(\theta), \dot{q}(\theta)) \\ = \frac{1-\alpha}{t-\theta} \partial_3 L(\theta, q(\theta), \dot{q}(\theta)). \end{aligned} \quad (1)$$

In this work we begin by generalizing the Euler-Lagrange equations (1) for FALVA problems with higher-order derivatives.

3 Main Results

In §3.1 and §3.2 we study FALVA problems with higher-order derivatives. The results are: Euler-Lagrange equations (Theorem 5) and a DuBois-Reymond stationary condition (Theorem 10) for such problems. Then, on section §3.3, we introduce the two-time optimal control FALVA problem, obtaining more general stationary conditions (Theorems 16 and 19).

3.1 Euler-Lagrange equations for higher-order FALVA problems

We prove Euler-Lagrange equations to higher-order problems of the calculus of variations with fractional integrals of Riemann-Liouville, i.e. to FALVA problems with higher-order derivatives.

Problem 3 *The higher-order FALVA problem consists to find stationary values of an integral functional*

$$I^m[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta))(t-\theta)^{\alpha-1} d\theta, \quad (P_m)$$

$m \geq 1$, subject to initial conditions

$$q^{(i)}(a) = q_a^i, \quad i = 0, \dots, m-1, \quad (2)$$

where $q^{(0)}(\theta) = q(\theta)$, $q^{(i)}(\theta)$ is the i -th derivative, $i \geq 1$; Γ is the Euler gamma function; $0 < \alpha \leq 1$; θ is the intrinsic time; t the observer's time, $t \neq \theta$; and the Lagrangian $L : [a, b] \times \mathbb{R}^{n \times (m+1)} \rightarrow \mathbb{R}$ is a function of class C^{2m} with respect to all the arguments.

Remark 4 *In the particular case when $m = 1$, functional (P_m) reduces to (P_1) and Problem 3 to 1.*

To establish the Euler-Lagrange stationary condition for Problem (P_m) , we follow the standard steps used to derive the necessary conditions in the calculus of variations.

Let us suppose $q(\cdot)$ a solution to Problem 3. The variation $\delta I^m[q(\cdot)]$ of the integral functional (P_m) is given by

$$\frac{1}{\Gamma(\alpha)} \int_a^t \left(\sum_{i=0}^m \partial_{i+2} L \cdot \delta q^{(i)} \right) (t-\theta)^{\alpha-1} d\theta, \quad (3)$$

where $\delta q^{(i)} \in C^{2m}([a, b]; \mathbb{R}^n)$ represents the variation of $q^{(i)}$, $i = 1, \dots, m$, and satisfy

$$\delta q^{(i)}(a) = 0. \quad (4)$$

Having in account conditions (4), repeated integration by parts of each integral containing $\delta q^{(i)}$ in (3) leads to

$$\begin{aligned} m = 1: \quad \delta I[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t \left[\left(\partial_2 L - \frac{d}{d\theta} \partial_3 L \right) \right. \\ \left. - \frac{1-\alpha}{t-\theta} \partial_3 L \right] (t-\theta)^{\alpha-1} \cdot \delta q d\theta; \end{aligned} \quad (5)$$

$$\begin{aligned} m = 2: \quad \delta I^2[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t \left[\left(\partial_2 L - \frac{d}{d\theta} \partial_3 L \right) \right. \\ \left. + \frac{d^2}{d\theta^2} \partial_4 L \right] \left(\frac{1-\alpha}{t-\theta} \left(\partial_3 L - 2 \frac{d}{d\theta} \partial_4 L \right) \right. \\ \left. - \frac{(1-\alpha)(2-\alpha)}{(t-\theta)^2} \partial_4 L \right) (t-\theta)^{\alpha-1} \cdot \delta q d\theta; \end{aligned} \quad (6)$$

and, in general,

$$\begin{aligned} \delta I^m[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t \left[\left(\partial_2 L + \sum_{i=1}^m (-1)^i \frac{d^i}{d\theta^i} \partial_{i+2} L \right) \right. \\ \left. - \frac{1-\alpha}{t-\theta} \sum_{i=1}^m i (-1)^{i-1} \frac{d^{i-1}}{d\theta^{i-1}} \partial_{i+2} L \right. \\ \left. - \sum_{k=2}^m \sum_{i=2}^k (-1)^{i-1} \frac{\Gamma(i-\alpha+1)}{(t-\theta)^i \Gamma(1-\alpha)} \binom{k}{k-i} \frac{d^{k-i}}{d\theta^{k-i}} \partial_{k+2} L \right] \\ \cdot (t-\theta)^{\alpha-1} \cdot \delta q d\theta. \end{aligned}$$

The integral functional $I^m[\cdot]$ has, by hypothesis, a stationary value for $q(\cdot)$, so that

$$\delta I^m[q(\cdot)] = 0.$$

The fundamental lemma of the calculus of variations asserts that all the coefficients of δq must vanish.

Theorem 5 (higher-order Euler-Lagrange equations) If $q(\cdot)$ gives a stationary value to functional (P_m) , then $q(\cdot)$ satisfy the higher-order Euler-Lagrange equations

$$\sum_{i=0}^m (-1)^i \frac{d^i}{d\theta^i} \partial_{i+2} L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) = F \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(2m-1)}(\theta) \right), \quad (7)$$

where $m \geq 1$ and

$$\begin{aligned} F \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(2m-1)}(\theta) \right) &= \frac{1-\alpha}{t-\theta} \sum_{i=1}^m i (-1)^{i-1} \frac{d^{i-1}}{d\theta^{i-1}} \partial_{i+2} L \\ + \sum_{k=2}^m \sum_{i=2}^k (-1)^{i-1} \frac{\Gamma(i-\alpha+1)}{(t-\theta)^i \Gamma(1-\alpha)} \binom{k}{k-i} \frac{d^{k-i}}{d\theta^{k-i}} \partial_{k+2} L \end{aligned}$$

with the partial derivatives of the Lagrangian L evaluated at $(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta))$.

Remark 6 Function F in (7) may be viewed as an external non-conservative friction force acting on the system. If $\alpha = 1$, then $F = 0$ and equation (7) is nothing more than the standard Euler-Lagrange equation for the classical problem of the calculus of variations with higher-order derivatives:

$$\sum_{i=0}^m (-1)^i \frac{d^i}{d\theta^i} \partial_{i+2} L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) = 0.$$

Remark 7 If $m = 1$, the Euler-Lagrange equations (7) coincide with the Euler-Lagrange equations (1).

Remark 8 For $m = 2$, the Euler-Lagrange equations (7) reduce to

$$\begin{aligned} \left(\partial_2 L(\theta, q, \dot{q}, \ddot{q}) - \frac{d}{d\theta} \partial_3 L(\theta, q, \dot{q}, \ddot{q}) \right. \\ \left. + \frac{d^2}{d\theta^2} \partial_4 L(\theta, q, \dot{q}, \ddot{q}) \right) = F(\theta, q, \dot{q}, \ddot{q}, \ddot{\ddot{q}}) \quad (9) \end{aligned}$$

where

$$\begin{aligned} F(\theta, q, \dot{q}, \ddot{q}, \ddot{\ddot{q}}) &= \frac{1-\alpha}{t-\theta} \left(\partial_3 L(\theta, q, \dot{q}, \ddot{q}) \right. \\ &\quad \left. - 2 \frac{d}{d\theta} \partial_4 L(\theta, q, \dot{q}, \ddot{q}) \right) \\ &\quad - \frac{\Gamma(3-\alpha)}{(t-\theta)^2 \Gamma(1-\alpha)} \binom{2}{0} \partial_4 L(\theta, q, \dot{q}, \ddot{q}) \\ &= \frac{1-\alpha}{t-\theta} \left(\partial_3 L(\theta, q, \dot{q}, \ddot{q}) - 2 \frac{d}{d\theta} \partial_4 L(\theta, q, \dot{q}, \ddot{q}) \right) \\ &\quad - \frac{(1-\alpha)(2-\alpha)}{(t-\theta)^2} \partial_4 L(\theta, q, \dot{q}, \ddot{q}). \quad (10) \end{aligned}$$

Proof: Theorem 5 is proved by induction. For $m = 1$ and $m = 2$, the Euler-Lagrange equations (1) and (9)–(10) are obtained applying the fundamental lemma of the calculus of variations respectively to (5) and (6). From the induction hypothesis,

$$\begin{aligned} \sum_{i=0}^j (-1)^i \frac{d^i}{d\theta^i} \partial_{i+2} L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ = F \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(2j-1)}(\theta) \right), \quad m = j > 2. \end{aligned} \quad (11)$$

We need to prove that equations (7)–(8) hold for $m = j + 1$. For simplicity, let us focus our attention on the variation of $q^{(j+1)}(\theta)$. From hypotheses (variation of $q^{(j+1)}(\theta)$ up to order $m = j$), and having in mind that $C_i^j + C_{i+1}^j = C_{i+1}^{j+1}$ and $m\Gamma(m) = \Gamma(m+1)$, we obtain equations (7) for $m = j+1$ using integration by parts followed by the application of the fundamental lemma of the calculus of variations. \square

It is convenient to introduce the following quantity (cf. [15]):

$$\psi^j = \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{d\theta^i} \partial_{i+j+2} L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right), \quad (12)$$

$j = 1, \dots, m$. This notation is useful for our purposes because of the following property:

$$\frac{d}{d\theta} \psi^j = \partial_{j+1} L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \psi^{j-1}, \quad (13)$$

$j = 1, \dots, m$.

Remark 9 Equation (7) can be written in the following form:

$$\begin{aligned} \partial_2 L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \frac{d}{d\theta} \psi^1 \\ = F \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(2m-1)}(\theta) \right). \quad (14) \end{aligned}$$

3.2 DuBois-Reymond condition for higher-order FALVA problems

We now prove a DuBois-Reymond condition for FALVA problems. The result seems to be new even for $m = 1$ (Corollary 12).

Theorem 10 (higher-order DuBois-Reymond condition) A necessary condition for $q(\cdot)$ to be a solution to Problem 3 is given by the following higher-order

DuBois-Reymond condition:

$$\begin{aligned} \frac{d}{d\theta} \left\{ L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \sum_{j=1}^m \psi^j \cdot q^{(j)}(\theta) \right\} \\ = \partial_1 L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ + F \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(2m-1)}(\theta) \right) \cdot \dot{q}(\theta), \quad (15) \end{aligned}$$

where F and ψ^j are defined by (8) and (12), respectively.

Remark 11 If $\alpha = 1$, then $F = 0$ and condition (15) is reduced to the classical higher-order DuBois-Reymond condition (see e.g. [15]):

$$\begin{aligned} \partial_1 L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ = \frac{d}{d\theta} \left\{ L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \sum_{j=1}^m \psi^j \cdot q^{(j)}(\theta) \right\} \end{aligned}$$

Proof: The total derivative of

$$L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \sum_{j=1}^m \psi^j \cdot q^{(j)}(\theta)$$

with respect to θ is:

$$\begin{aligned} \frac{d}{d\theta} \left\{ L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \sum_{j=1}^m \psi^j \cdot q^{(j)}(\theta) \right\} \\ = \frac{\partial L}{\partial \theta} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ + \sum_{j=0}^m \frac{\partial L}{\partial q^{(j)}} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \cdot q^{(j+1)}(\theta) \\ - \sum_{j=1}^m \left(\dot{\psi}^j \cdot q^{(j)}(\theta) + \psi^j \cdot q^{(j+1)}(\theta) \right). \quad (16) \end{aligned}$$

From (13) it follows that (16) is equivalent to

$$\begin{aligned} \frac{d}{d\theta} \left\{ L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \sum_{j=1}^m \psi^j \cdot q^{(j)}(\theta) \right\} \\ = \frac{\partial L}{\partial \theta} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ + \sum_{j=0}^m \frac{\partial L}{\partial q^{(j)}} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \cdot q^{(j+1)}(\theta) \\ - \sum_{j=1}^m \left[\left(\frac{\partial L}{\partial q^{(j-1)}} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \right. \right. \\ \left. \left. - \psi^{j-1} \right) \cdot q^{(j)}(\theta) + \psi^j \cdot q^{(j+1)}(\theta) \right]. \quad (17) \end{aligned}$$

We now simplify the last term on the right-hand side of (17):

$$\begin{aligned} \sum_{j=1}^m \left[\left(\frac{\partial L}{\partial q^{(j-1)}} - \psi^{j-1} \right) \cdot q^{(j)}(\theta) + \psi^j \cdot q^{(j+1)}(\theta) \right] \\ = \sum_{j=0}^{m-1} \left[\frac{\partial L}{\partial q^{(j)}} \cdot q^{(j+1)}(\theta) - \psi^j \cdot q^{(j+1)}(\theta) \right. \\ \left. + \psi^{j+1} \cdot q^{(j+2)}(\theta) \right] \\ = \sum_{j=0}^{m-1} \left[\frac{\partial L}{\partial q^{(j)}} \cdot q^{(j+1)}(\theta) \right] - \psi^0 \cdot \dot{q}(\theta) + \psi^m \cdot q^{(m+1)}(\theta), \quad (18) \end{aligned}$$

where the partial derivatives of the Lagrangian L are evaluated at $(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta))$. Substituting (18) into (17), and using the higher-order Euler-Lagrange equations (7), we obtain the intended result, that is,

$$\begin{aligned} \frac{d}{d\theta} \left\{ L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) - \sum_{j=1}^m \psi^j \cdot q^{(j)}(\theta) \right\} \\ = \frac{\partial L}{\partial \theta} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ + \frac{\partial L}{\partial q^{(m)}} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \cdot q^{(m+1)}(\theta) \\ + \psi^0 \cdot \dot{q}(\theta) - \psi^m \cdot q^{(m+1)}(\theta) \\ = \partial_1 L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right) \\ + F \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(2m-1)}(\theta) \right) \cdot \dot{q}(\theta), \end{aligned}$$

since, by definition,

$$\psi^m = \frac{\partial L}{\partial q^{(m)}} \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right)$$

and

$$\psi^0 = \sum_{i=0}^m (-1)^i \frac{d^i}{d\theta^i} \partial_{i+2} L \left(\theta, q(\theta), \dot{q}(\theta), \dots, q^{(m)}(\theta) \right).$$

□

Corollary 12 (DuBois-Reymond condition) If $q(\cdot)$ is a solution of Problem 1, then the following (first-order) DuBois-Reymond condition holds:

$$\begin{aligned} \frac{d}{d\theta} \{ L(\theta, q(\theta), \dot{q}(\theta)) - \partial_3 L(\theta, q(\theta), \dot{q}(\theta)) \cdot \dot{q}(\theta) \} \\ = \partial_1 L(\theta, q(\theta), \dot{q}(\theta)) + \frac{1-\alpha}{t-\theta} \partial_3 L(\theta, q(\theta), \dot{q}(\theta)) \cdot \dot{q}(\theta). \quad (19) \end{aligned}$$

Proof: For $m = 1$, condition (15) is reduced to

$$\begin{aligned} \frac{d}{d\theta} \{L(\theta, q(\theta), \dot{q}(\theta)) - \psi^1 \cdot \dot{q}(\theta)\} \\ = \partial_1 L(\theta, q(\theta), \dot{q}(\theta)) + F(\theta, q(\theta), \dot{q}(\theta)) \cdot \dot{q}(\theta). \end{aligned} \quad (20)$$

Having in mind (8) and (12), we obtain that

$$\psi^1 = \partial_3 L(\theta, q(\theta), \dot{q}(\theta)), \quad (21)$$

$$F(\theta, q(\theta), \dot{q}(\theta)) = \frac{1-\alpha}{t-\theta} \partial_3 L(\theta, q(\theta), \dot{q}(\theta)). \quad (22)$$

One finds the intended equality (19) by substituting the quantities (21) and (22) into (20). \square

3.3 Stationary conditions for optimal control FALVA problems

Fractional optimal control problems have been studied in [2, 8, 9]. Here we obtain stationary conditions for two-time FALVA problems of optimal control. We begin by defining the problem.

Problem 13 *The two-time optimal control FALVA problem consists in finding the stationary values of the integral functional*

$$I[q(\cdot), u(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\theta, q(\theta), u(\theta)) (t-\theta)^{\alpha-1} d\theta, \quad (23)$$

when subject to the control system

$$\dot{q}(\theta) = \varphi(\theta, q(\theta), u(\theta)) \quad (24)$$

and the initial condition $q(a) = q_a$. The Lagrangian $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$ and the velocity vector $\varphi : [a, b] \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$ are assumed to be C^1 functions with respect to all their arguments. In accordance with the calculus of variations, we suppose that the control functions $u(\cdot)$ take values on an open set of \mathbb{R}^r .

Remark 14 *Problem 1 is a particular case of Problem 13 where $\varphi(\theta, q, u) = u$. FALVA problems of the calculus of variations with higher-order derivatives are also easily written in the optimal control form (23)-(24). For example, the integral functional of the second-order FALVA problem of the calculus of variations,*

$$I^2[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^t L(\theta, q(\theta), \dot{q}(\theta), \ddot{q}(\theta)) (t-\theta)^{\alpha-1} d\theta,$$

is equivalent to

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_a^t L(\theta, q^0(\theta), q^1(\theta), u(\theta)) (t-\theta)^{\alpha-1} d\theta, \\ \begin{cases} \dot{q}^0(\theta) = q^1(\theta), \\ \dot{q}^1(\theta) = u(\theta). \end{cases} \end{aligned}$$

We now adopt the Hamiltonian formalism. We reduce (23)-(24) to the form (P_1) by considering the augmented functional:

$$\begin{aligned} J[q(\cdot), u(\cdot), p(\cdot)] \\ = \frac{1}{\Gamma(\alpha)} \int_a^t [\mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)) - p(\theta) \cdot \dot{q}(\theta)] d\theta, \end{aligned} \quad (25)$$

where the Hamiltonian \mathcal{H} is defined by

$$\mathcal{H}(\theta, q, u, p) = L(\theta, q, u) (t-\theta)^{\alpha-1} + p \cdot \varphi(\theta, q, u). \quad (26)$$

Definition 15 (Process) *A pair $(q(\cdot), u(\cdot))$ that satisfies the control system $\dot{q}(\theta) = \varphi(\theta, q(\theta), u(\theta))$ and the initial condition $q(a) = q_a$ of Problem 13 is said to be a process.*

Next theorem gives the weak Pontryagin maximum principle for Problem 13.

Theorem 16 *If $(q(\cdot), u(\cdot))$ is a stationary process for Problem 13, then there exists a vectorial function $p(\cdot) \in C^1([a, b]; \mathbb{R}^n)$ such that for all θ the tuple $(q(\cdot), u(\cdot), p(\cdot))$ satisfy the following conditions:*

- the Hamiltonian system

$$\begin{cases} \dot{q}(\theta) = \partial_4 \mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)), \\ \dot{p}(\theta) = -\partial_2 \mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)); \end{cases} \quad (27)$$

- the stationary condition

$$\partial_3 \mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)) = 0; \quad (28)$$

where \mathcal{H} is given by (26).

Proof: We begin by remarking that the first equation in the Hamiltonian system, $\dot{q} = \partial_4 \mathcal{H}$, is nothing more than the control system (24). We write the augmented functional (25) in the following form:

$$\frac{1}{\Gamma(\alpha)} \int_a^t \left[\frac{\mathcal{H} - p(\theta) \cdot \dot{q}(\theta)}{(t-\theta)^{\alpha-1}} \right] (t-\theta)^{\alpha-1} d\theta, \quad (29)$$

where \mathcal{H} is evaluated at $(\theta, q(\theta), u(\theta), p(\theta))$. Intended conditions are obtained by applying the stationary condition (1) to (29):

$$\begin{cases} \frac{d}{d\theta} \frac{\partial}{\partial \dot{q}} \left[\frac{\mathcal{H}-p\dot{q}}{(t-\theta)^{\alpha-1}} \right] = \frac{\partial}{\partial q} \left[\frac{\mathcal{H}-p\dot{q}}{(t-\theta)^{\alpha-1}} \right] - \frac{1-\alpha}{t-\theta} \frac{\partial}{\partial \dot{q}} \left[\frac{\mathcal{H}-p\dot{q}}{(t-\theta)^{\alpha-1}} \right] \\ \frac{d}{d\theta} \frac{\partial}{\partial \dot{u}} \left[\frac{\mathcal{H}-p\dot{q}}{(t-\theta)^{\alpha-1}} \right] = \frac{\partial}{\partial u} \left[\frac{\mathcal{H}-p\dot{q}}{(t-\theta)^{\alpha-1}} \right] - \frac{1-\alpha}{t-\theta} \frac{\partial}{\partial \dot{u}} \left[\frac{\mathcal{H}-p\dot{q}}{(t-\theta)^{\alpha-1}} \right] \end{cases}$$

$$\Leftrightarrow \begin{cases} -\dot{p} = \partial_2 \mathcal{H} \\ 0 = \partial_3 \mathcal{H} \end{cases}$$

□

Remark 17 For FALVA problems of the calculus of variations, Theorem 16 takes the form of Theorem 5.

Definition 18 (Pontryagin FALVA extremal) We call any tuple $(q(\cdot), u(\cdot), p(\cdot))$ satisfying Theorem 16 a Pontryagin FALVA extremal.

Next theorem generalizes the DuBois-Reymond condition (15) to Problem 13.

Theorem 19 The following property holds along the Pontryagin FALVA extremals:

$$\frac{d\mathcal{H}}{d\theta}(\theta, q(\theta), u(\theta), p(\theta)) = \partial_1 \mathcal{H}(\theta, q(\theta), u(\theta), p(\theta)). \quad (30)$$

Proof: Equality (30) is a simple consequence of Theorem 16. □

Remark 20 In the classical framework, i.e. for $\alpha = 1$, the Hamiltonian \mathcal{H} does not depend explicitly on θ when the Lagrangian L and the velocity vector φ are autonomous. In that case, it follows from (30) that the Hamiltonian \mathcal{H} (interpreted as energy in mechanics) is conserved. In the FALVA setting, i.e. for $\alpha \neq 1$, this is no longer true: equality (30) holds but we have no conservation of energy since, by definition (cf. (26)), the Hamiltonian \mathcal{H} is never autonomous (\mathcal{H} always depend explicitly on θ for $\alpha \neq 1$, thus $\partial_1 \mathcal{H} \neq 0$).

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